ON THE CONVERSE OF ROUTH'S THEOREM

(OB OBRASHCHENII TEOREMY RAUSA)

PMM Vol.27, No.5, 1963, pp. 890-893

VAN CHZHAO-LIN
(Moscow)

(Received April 26, 1963)

In works dealing with applications of Chetaev's method, the question of the converse of Routh's theorem is considered [1,2]. When applied to the equations of motion in normal coordinates, one may prove instability theorems for systems which are gyroscopically constrained systems [3, 4,5].

1. Suppose that $q_1, \ldots, q_n$ are independent Lagrangean coordinates of a certain holonomic conservative mechanical system, having constraints which do not depend explicitly upon time. Let T be the kinetic energy, and U be the potential function of the system. Let us suppose that the coordinates $q_{m+1}, \ldots, q_n$ $(m < n)$ are cyclic, in the sense that

$$\partial L / \partial q_\alpha = 0 \qquad (\alpha = m+1, \ldots, n)$$

where $L = T + U$ is the Lagrangean of the system. The equations of motion of such a system

$$\frac{d}{dt}\frac{\partial L}{\partial q_j'} - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, \ldots, n)$$

possess $n - m$ first integrals

$$\partial L / \partial q_\alpha' = \beta_\alpha$$

where the β_α are constants of integration. In the case under consideration, the Lagrangean equations, for the noncyclic coordinates, have the form [3]

$$\frac{d}{dt}\frac{\partial R}{\partial q_i'} - \frac{\partial R}{\partial q_i} = 0 \qquad (i = 1, \ldots, m) \qquad R = L - \sum_\alpha q_\alpha' \beta_\alpha \tag{1.1}$$

The function R does not depend on the cyclic coordinates or on their velocities; it has the form

$$R = R_2 + R_1 + R_0$$

where R_0, the perturbed potential function, does not depend on the noncyclic velocities, R_1 is a linear form in the acyclic velocities, and R_2 is a positive definite quadratic form in the acyclic velocities.

Suppose that, for certain values of β_α, the equations of motion (1.1) possess the particular solution $q_i = 0$. This solution corresponds to a stationary motion in which a single cyclic coordinate q_α has been altered.

Let us suppose, without loss of generality, that for $q_i = 0$ the function R_0 has the value zero. In this manner, for fixed values of the β_α, the equations of the perturbed motion are equations (1.1).

In view of Routh's theorem [1], the stationary motion corresponding to a particular solution $q_i = 0$ is stable, provided that the perturbed potential function R_0 is a strict maximum.

Under certain conditions, this theorem admits a converse. The equations of motion (1.1) may be written in the form

$$\frac{d}{dt}\frac{\partial R_2}{\partial q_i'} - \frac{\partial R_2}{\partial q_i} = \frac{\partial R_0}{\partial q_i} + \Gamma_i \qquad (1.2)$$

where the Γ_i are the gyroscopic forces

$$R_2 = \frac{1}{2}\sum_{s,r=1}^{m} a_{sr}(q_1, \ldots q_m)\, q_s' q_r' \qquad (a_{sr} = a_{rs})$$

$$\Gamma_i = -\frac{d}{dt}\frac{\partial R_1}{\partial q_i'} + \frac{\partial R_1}{\partial q_i} = \sum_{j=1}^{m} g_{ji}(q_1, \ldots, q_m)\, q_j' \qquad (g_{ji} = -g_{ij};\ g_{ii} = 0)$$

1. Consider the gyroscopically unconstrained system, in which $g_{ij} = 0$ for $i, j = 1, \ldots, m$, that is, $\Gamma_i = 0$, in particular, this occurs when $R_1 \equiv 0$. Then the equations of the perturbed motion become

$$\frac{d}{dt}\frac{\partial R^*}{\partial q_i'} - \frac{\partial R^*}{\partial q_i} = 0$$

Here, $R^* = R_2 + R_0$, that is, it has the same form as the equations of a perturbed motion in the neighborhood of an equilibrium position. If we suppose that

$$p_i = \partial R^* / \partial q_i', \qquad H^* = -R^* + \sum_i \frac{\partial R^*}{\partial q_i'} q_i'$$

then we obtain the canonical Hamilton's equations for acyclic coordinates

$$dq_i / dt = \partial H^* / \partial p_i, \qquad dp_i / dt = - \partial H^* / \partial q_i \qquad (i = 1, \ldots, m) \tag{1.3}$$

Consider the Hamilton-Jacobi equation corresponding to (1.3)

$$H^* \left(q_1, \ldots, q_m; \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_m} \right) = h \tag{1.4}$$

If the complete integral of equation (1.4) is

$$W = W(q_1, \ldots, q_m; \alpha_2, \ldots, \alpha_m, h)$$

then the momenta p_i are given, in the perturbed motion, according to Jacobi's theorem, by the formulas

$$p_i = \partial W / \partial q_i \qquad (i = 1, \ldots, m)$$

where α_s, h are arbitrary constants, for $s = 2, \ldots, m$.

In the case of equilibrium, Chetaev's theorem [2] is valid. When applied to the problem under consideration, this theorem may be formulated as follows.

Theorem 1. If, for the isolated stationary motion, the function R_0, supposed to be an analytic function, is not a maximum, then the stationary motion of the gyroscopically unconstrained system, $q_i = 0$, is unstable.

2. Consider the equations of the perturbed motion, (1.1). Let us suppose that

$$R_1 = \sum_i c_i (q_1, \ldots, q_m)\, q_i' = \sum_i c_i^{(0)} q_i' + \sum_i c_i^{(1)} q_i' + \sum_i c_i^{(2)} q_i' + Q_1$$

$$c_i^{(0)} = c_i(0, \ldots, 0), \quad c_i^{(1)} = \sum_k \left(\frac{\partial c_i}{\partial q_k} \right)_0 q_k, \qquad c_i^{(2)} = \frac{1}{2} \sum_{s,r} \left(\frac{\partial^2 c_i}{\partial q_s \partial q_r} \right)_0 q_s q_r$$

where the term Q_1 is of not less than the fourth order. Equations (1.1) may be written in the following form:

$$\frac{d}{dt} \frac{\partial R^*}{\partial q_i'} - \frac{\partial R^*}{\partial q_i} = 0 \qquad (i = 1, \ldots, m) \tag{1.5}$$

$$R^* = R_2 + R_1^* + R_0$$

$$R_1^* = \frac{1}{2} \sum_{ij} \left(\frac{\partial c_i^{(1)}}{\partial q_j} - \frac{\partial c_j^{(1)}}{\partial q_i} \right) q_i' q_j + \sum_i c_i^{(2)} q_i' + Q_1 = \frac{1}{2} \sum_{ij} g_{ij}^{(0)} q_i' q_j + \sum_i c_i^{(2)} q_i' + Q_1$$

Consequently, by Chetaev's method [2], one may prove the following theorem for gyroscopically constrained systems, and for sufficiently small (in absolute value) values of the variables q_i, q_i' ($i = 1,\ldots,m$).

Theorem 2. If, for an isolated stationary motion, the function R_0, assumed to be analytic, is not a maximum; and if the coefficients $g_{ij}^{(0)}$ in the expansion of the function R_1* are all zero, then the stationary motion is unstable.

Indeed, in view of the hypotheses of the theorem, in an arbitrary domain $R_0 > R_0^* > 0$, for any R_0^*, and corresponding to any point of the region C defined by the inequalities $R_0 > 0$ and $q_1^2 + \ldots + q_m^2 < l$, there is no movable singularity of the complete integral W; further, by a suitable choice of the initial conditions we obtain

$$dW / dt = \sum_i \frac{\partial W}{\partial q_i} q_i' = \sum_i p_i q_i' = \sum_i \frac{\partial R^*}{\partial q_i'} q_i' = 2R_2 + Q_2 > 0$$

where Q_2 contains terms of order not less than the third order of smallness, depending on the velocities q_i'.

Consequently, the function W satisfies all the hypotheses of Chetaev's theorem on instability in the domain $R_0 > R_0^* > 0$ inside the region C. The assertion has thus been proved.

2. Consider a gyroscopically constrained system in normal coordinates, and introduce the notation

$$a_{sr}^0 = a_{sr}(0, \ldots, 0), \qquad g_{ij}^0 = g_{ij}(0, \ldots, 0) \qquad (g_{ij}^0 = -\overset{\circ}{g}_{ji};\ \overset{\circ}{g}_{ii} = 0)$$

$$b_{ij} = \left(\frac{\partial^2 R_0}{\partial q_i \partial q_j}\right)_0 \qquad (b_{ij} = b_{ji})$$

The equations of the perturbed motion, (1.2), may be reduced to the form [4,6]

$$\sum_{j=1}^{m} \left(a_{ij}^0 \frac{dq_j'}{dt} + g_{ij}^0 q_j' + b_{ij} q_j\right) + Q_i = 0 \qquad (i = 1, \ldots, m) \tag{2.1}$$

where the Q_i are holomorphic functions, containing terms of not less than the second order.

It is known [4] that there exists a nonsingular linear transformation with constant coefficient which reduces the first approximation to equations (2.1) to the form

$$x_i'' + \sum_j g_{ij}^* x_j' + \lambda_i x_i = 0 \qquad (i = 1, \ldots, m) \tag{2.2}$$

where x_i are normal coordinates, λ_i are the stability coefficients of Poincaré independent of the gyroscopic forces, and g_{ij}* are constants having properties similar to the g_{ij}°.

Let us consider the case when some of the stability coefficients of Poincaré, λ_i, are zero.

Suppose that the skew symmetric matrix, whose determinant is formed from the gyroscopic terms $g_{\nu\mu}^*(\nu, \mu = 1, \ldots, s)$, has the form $\| g_{\nu\mu}^* \|_1^s$.

Theorem 3. If, in equations (2.2), $\lambda_k = 0$ $(k = 1, \ldots, s < m)$, and the determinant of the skew symmetric matrix $\| g_{\nu\mu}^* \|_1^s$ is different from zero so that the degree of instability is odd, then the stationary motion is unstable.

Proof. Since for equations (2.2) among the roots of the characteristic equation one finds $\lambda_k = 0$, one must have

$$\Delta(\lambda) = (\lambda^s) \begin{vmatrix} \lambda & g_{12}^* & \ldots & g_{1s}^* & \lambda g_{1,s+1}^* & \ldots & \lambda g_{1m}^* \\ g_{21}^* & \lambda & \ldots & g_{2s}^* & \lambda g_{2,s+1}^* & \ldots & \lambda g_{2m}^* \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{s1}^* & g_{s2}^* & \ldots & \lambda & \lambda g_{s,s+1}^* & \ldots & \lambda g_{sm}^* \\ g_{s+1,1}^* & g_{s+1,2}^* & \ldots & g_{s+1,s}^* & \lambda^2 + \lambda_{s+1} & \ldots & \lambda g_{s+1,m}^* \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{m1}^* & g_{m2}^* & \ldots & g_{ms}^* & \lambda g_{m,s+1}^* & \ldots & \lambda^2 + \lambda m \end{vmatrix}$$

Hence, the expansion of equation $\Delta(\lambda) = 0$ may be written in the form

$$\Delta(\lambda) = (\lambda^s)(\lambda^{2m-s} - a_1\lambda^{2m-s-1} + \ldots + (-1)^{2m-s}a_m) = 0$$

where the constant term is

$$(-1)^{2m-s}a_m = (\lambda_{s+1} \ldots \lambda_m) \begin{vmatrix} 0 & g_{12}^* & \ldots & g_{1s}^* \\ g_{21}^* & 0 & \ldots & g_{2s}^* \\ \cdot & \cdot & \cdot & \cdot \\ g_{s1}^* & g_{s2}^* & \ldots & 0 \end{vmatrix}$$

It is known [4] that when the determinant of the skew symmetric matrix $\| g_{\nu\mu}^* \|_1^s$ is not zero, it must be positive, and thus s must be even.

Thus, in view of the hypotheses of the theorem, the product of the remaining $2m - s$ roots, a_m, must be negative. Thus, at least one of the non-zero characteristic roots must be positive.

Therefore, in the case under consideration, in view of Liapunov's theorem, the first approximation to the unperturbed motion of equation (2.1) must be unstable.

The theorem is proved.

Note 1. For an isolated position of equilibrium, Chetaev [3] proved a theorem on the instability of mechanical systems which are acted upon by potential and gyroscopic forces. This theorem may be considered as a certain sort of converse of Routh's theorem, and may be formulated as follows.

If, for equation (2.2), all λ_i are different from zero, and the degree of instability is odd, then the stationary motion is unstable.

Note 2. In [7, Section 5], a theorem on the instability of the stationary motion is proved under the hypothesis that

$$\left(\frac{\partial^r H}{\partial p_i \partial q_1^{\ m_1} \cdots \partial q_m^{\ m_k}}\right)^\circ = 0 \qquad (*)$$

for arbitrary $m_1 + m_2 + \ldots + m_k > 0$ and $i \leqslant k$, and that the function W_1, which depends only on the variations of the coordinates, is a homogeneous function.

If (*) holds, then the equations of the perturbed motion become

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{k}\left[\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)^\circ + \delta_{ij}\right]\eta_j, \qquad \frac{d\eta_i}{dt} = -\frac{\partial}{\partial \xi_i}\sum_{l,j=1}^{k}\delta_{lj}\eta_l\eta_j - \frac{\partial W_1}{\partial \xi_i}$$

where the $\delta_{ij}(\xi_1, \ldots, \xi_k)$ vanish when all the $\xi_1, \ldots, \xi_k$ are zero; it is readily seen that this case corresponds to the case in which $R_1 = 0$, that is to say that the system of equations of the perturbed motion is a gyroscopically unconstrained system.

On the other hand, the theorems on the instability of the stationary motion given by us (see Theorems 1 and 2 above) hold under more general hypotheses.

Note 3. It is to be noticed that [8] contains an attempt to prove a theorem for gyroscopic systems, under the hypotheses that

1) $$\partial (R_0 + U)\, q_i \,/\, \partial q_i > 0$$

2) the velocities $\eta_1, \ldots, \eta_s$ are constrained by the relations

$$P_{ik}\eta_k = 0 \qquad (i = 1, \ldots, 2p;\ k = 1, \ldots, s)$$

However, the author does not prove the inequality (1). Besides, he does not even prove that when the constraints (2) are added, the differential equations of motion remain unchanged. In the absence of proofs of these two assertions, the theorem of [8] must be regarded as without proof.

In conclusion, the author thanks V.V. Rumiantsev for attention given to this paper.

BIBLIOGRAPHY

1. Routh, The advanced part of a treatise on the dynamics of rigid bodies. 4th Ed., 1884.

2. Chetaev, N.G., O neustoichivosti ravnovesiia, kogda silovaia funktsiia ne est' maksimum (On the instability of equilibrium, when the force function is not a maximum). *Uchen. Zap. Kazan. Univ.*, Vol. 98, No. 9, 1938.

3. Chetaev, N.G., *Ustoichivosti dvizheniia* (*Stability of motion*). Gostekhteoretizdat, 1955.

4. Merkin, D.R., *Giroskopicheskie sistemy* (*Gyroscopic systems*). Gostekhteoretizdat, 1956.

5. Rumiantsev, V.V., Ob ustoichivosti ravnomernykh vrashchenii mekhanicheskikh sistem (On the stability of uniformly rotating mechanical systems). *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 6, 1962.

6. Matrosov, V.M., K voprosu ustoichivosti giroskopicheskikh sistem (On the question of the stability of gyroscopic systems). *Trudy KAI*, Vol. 49, *ser. Mekhanika i matematika*, 1959.

7. Pozharitskii, G.K., O neustanovivshemsia dvizhenii konservativnykh golonomnykh sistem (On unstable motions of conservative holonomic systems). *PMM* Vol. 20, No. 3, 1956.

8. Kharlamov, S.A., Ob obrashchenii odnoi teoremy Rausa (On the converse of a Routh theorem). *Vestn. MGU*, No. 6, 1961.

Translated by J.B.D.